Automatica, Vol. 24, No. 5, pp. 707–710, 1988
Printed in Great Britain.

0005-1098/88 \$3.00 + 0.00
Pergamon Press plc

Brief Paper

The Optimal Regulation of Generalized State-space Systems with Quadratic Cost* †

CHENG ZHAOLIN‡, HONG HUIMIN§ and ZHANG JIFENG||

Key Words—Linear optimal regulator; regulator theory; stability; linear systems; system theory; linear algebra.

Abstract—In this paper, the optimal feedback control for regulating generalized state-space systems with quadratic cost is presented by using an algebraic method, and the weighting matrix Q in the cost is allowed to be positive semi-definite.

1. *Introduction*

SINCE ROSENBROCK (1974) introduced the restricted system equivalence (RSE) for the generalized state-space system, i.e. for the linear system

$$E\dot{x} = Ax + Bu \tag{1}$$

with E being singular and $\det(sE - A) \not\equiv 0$, there has been a lot of research into various problems such as controllability and observability of the system (Campbell, 1980; Yip and Sincovec, 1981; Verghese *et al.*, 1981; Cobb, 1984; also see Lewis, 1986), poles assignment and the elimination of impulsive behavior of the system by state feedback (Cobb, 1981), and optimal regulators with quadratic cost, i.e. the LQ problem (Pandolfi, 1981; Cobb, 1983a; Bender and Laub, 1987a, b), and so on. However, the weighting matrices in the LQ problem investigated by Cobb (1983a) are positive definite. In this paper, the LQ problem is treated algebraically for the general case where Q is allowed to be positive semi-definite, and the problem is transformed into the LQ problem for a regular state-space system by invoking strong stabilizability and strong detectability of system (1). We believe that the algebraic method adopted here is easier than the geometric one used by Cobb (1983a) for engineeers to comprehend. The same problem has also been considered by Bender and Laub (1987a, b). They used Hamiltonian minimization. However, for the infinite-horizon case, stronger conditions are required there than that required in this paper.

* Received 22 January 1987; revised 4 December 1987; received in final form 23 January 1988. The original version of this paper was presented at the 10th IFAC World Congress which was held in Munich, F.R.G. during July 1987. The Published Proceedings of this IFAC Meeting may be ordered from: Pergamon Press plc, Headington Hill Hall, Oxford OX3 0BW, England. This paper was recommended for publication in revised form by Associate Editor A. K. Mahalanabis under the direction of Editor H. Kwakernaak.

† This project is supported by the Science Fund of the Chinese Academy of Sciences.

‡ The Department of Mathematics, Shandong University, Jinan, P.R.C.

§ The Department of Control Engineering, Shandong University, Weihai, P.R.C.

|| The Institute of Systems Science, Academia Sinica, Beijing, P.R.C.

2. *Optimal regulator with quadratic cost*

For system (1), with initial state $x(0^-)$ which can be consistent or not (Cobb, 1983b), consider the cost functional

$$J(u, x(0^-)) = \int_{0-}^{\infty} (x^{\tau}Qx + u^{\tau}Ru)\,dt \tag{2}$$

where u is the input, $u \in R^r$, x is the state, $x \in R^n$, Q and R are constant matrices, Q is positive semi-definite and R is positive definite. Let $\mathcal{U}$ be the set of admissible controls in which any admissible control is piecewise sufficiently smooth and makes $J(u, x(0^-))$ finite. Then the LQ problem is to find the optimal control $u^* \in \mathcal{U}$, such that

$$J(u^*, x(0^-)) = \min_{u \in \mathcal{U}} J(u, x(0^-)). \tag{3}$$

Definition 1. System (1) is strongly stabilizable if

$$\operatorname{rank}[sE - A \quad B] = n \tag{4}$$

for any complex s with non-negative real part and $s = \infty$.

Lemma 1. Let T be a non-singular matrix such that

$$T[sE - A \quad B] = \begin{bmatrix} sE_1 - A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \tag{5}$$

where E_1 has full row rank; then equation (4) holds at $s = \infty$ if and only if

$$\operatorname{rank}\begin{bmatrix} E_1 & 0 \\ A_2 & B_2 \end{bmatrix} = n. \tag{6}$$

The proof of the lemma was given by Verghese *et al.* (1979). Equation (4), holding at $s = \infty$, is a necessary and sufficient condition for eliminating impulsive behaviour of system (1) (Cobb, 1981).

Lemma 2. For system (1), there exists a state feedback matrix $K \in R^{r \times n}$ such that the system

$$E\dot{x} = (A + BK)x + Bv \tag{7}$$

is RSE to the system

$$\begin{cases} \dot{x}_1 = A_1x_1 + B_1v \\ 0 = x_2 + B_2v \end{cases} \tag{8}$$

if and only if equation (4) holds at $s = \infty$, where $x_1 \in R^{n_E}$, $n_E = \operatorname{rank} E$, $x_2 \in R^{n-n_E}$, $A_1 \in R^{n_E \times n_E}$, $B_1 \in R^{n_E \times r}$, $B_2 \in R^{(n-n_E)\times r}$.

By saying that system (7) is RSE to system (8), we mean that there exist two non-singular matrices $F, G \in R^{n \times n}$ such that

$$F[sE - (A + BK) \quad B]\begin{bmatrix} G & 0 \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} sI_{n_E} - A_1 & 0 & B_1 \\ 0 & -I_{n-n_E} & B_2 \end{bmatrix}.$$

Definition 2. The system

$$\Sigma_0: \begin{cases} E\dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (9)$$

is strongly detectable if

$$\text{rank}\begin{bmatrix} C \\ sE - A \end{bmatrix} = n \quad (10)$$

for any complex s with non-negative real part and $s = \infty$.

A criterion of testing that equation (10) holds at $s = \infty$ was given by Verghese *et al.* (1981); see Lemma 3.

Lemma 3. Equation (10) holds at $s = \infty$ for system Σ_0 if and only if there are no constant vectors α, $\beta \in R^n$, $\beta \neq 0$, such that

$$\begin{bmatrix} C \\ sE - A \end{bmatrix}\beta = \begin{bmatrix} 0 \\ E\alpha \end{bmatrix}, \quad \text{for any } s \in C. \quad (11)$$

Decompose matrix Q of cost (2) in different ways: $Q = C^\tau C = C_1^\tau C_1$, and denote

$$\Sigma: \begin{cases} E\dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (12)$$

$$\Sigma_1: \begin{cases} E\dot{x} = AX + Bu \\ y = C_1 x. \end{cases} \quad (13)$$

Then we have the following lemma.

Lemma 4. Σ is strongly detectable if and only if Σ_1 is.

The lemma shows that the strong detectability for system Σ is independent of the decomposition of matrix Q. We omit its proof and proceed directly to the LQ problem for system (1).

Theorem. If system Σ is strongly stabilizable and strongly detectable, then the LQ problem for system (1) has a solution, the optimal control is realized by a linear state feedback when $t > 0$, and the optimal closed loop system is asymptotically stable.

Proof. The basic idea of the proof is to transfer the LQ problem of a generalized state-space system into that of a regular system by means of conceptions of strong stabilizability and strong detectability.

(a) *Reforming* $J(u, x(0^-))$

In view of the strong stabilizability of system Σ, from Lemma 2 we can find a matrix $K_1 \in R^{r\times n}$ such that the system

$$E\dot{x} = (A + BK_1)x + BV \quad (14)$$

is RSE to the system

$$\dot{x}_1 = A_1 x_1 + B_1 v \quad (15)$$

$$0 = x_2 + B_2 v \quad (16)$$

i.e. there exist non-singular matrices F, $G \in R^{n\times n}$ such that

$$x = G\bar{x} \quad (17)$$

$$FEG = \begin{bmatrix} I_{n_E} & 0 \\ 0 & 0 \end{bmatrix} \quad (18)$$

$$F(A + BK_1)G = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-n_E} \end{bmatrix} \quad (19)$$

$$FB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (20)$$

where $n_E = \text{rank } E$, $B_1 \in R^{n_E\times r}$, $B_2 \in R^{(n-n_E)\times r}$, $A_1 \in R^{n_E\times n_E}$, $\bar{x} = [x_1^\tau x_2^\tau]^\tau$, $x_1 \in R^{n_E}$, $x_2 \in R^{n-n_E}$. Let $x(t)$ be the solution of system (1) driven by $u \in \mathcal{U}$ with initial state $x(0^-)$. Set

$$v(t) = u(t) - K_1 x(t). \quad (21)$$

Then $x(t)$ satisfies (14), since $\bar{x}(t) = G^{-1}x(t)$, then components $x_1(t)$ and $x_2(t)$ of $\bar{x}(t)$ satisfy (15) and (16), respectively. Rewriting $J(u, x(0^-))$ via (16)–(21), we have

$$J(u, x(0^-)) = \int_{0^-}^{\infty} \begin{bmatrix} x_1 \\ v \end{bmatrix}^\tau \begin{bmatrix} I_{n_E} & 0 \\ 0 & -B_2 \\ 0 & I_r \end{bmatrix}^\tau \begin{bmatrix} G^\tau & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} I_n & K_1^\tau \\ 0 & I_r \end{bmatrix}$$

$$\times \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} I_n & 0 \\ K_1 & I_r \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} I_{n_E} & 0 \\ 0 & -B_2 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} x_1 \\ v \end{bmatrix} \mathrm{d}t. \quad (22)$$

Representing the weighting matrix in (22) by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\tau & M_{22} \end{bmatrix} \quad (23)$$

where $M_{11} \in R^{n_E\times n_E}$, $M_{12} \in R^{n_E\times r}$, $M_{22} \in R^{r\times r}$, we have

$$J(u, x(0^-)) = \int_{0^-}^{\infty} \begin{bmatrix} x_1 \\ v \end{bmatrix}^\tau \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\tau & M_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ v \end{bmatrix} \mathrm{d}t. \quad (24)$$

(b) *Showing* M_{22} *positive definite and rewriting cost* (24).

Assume the contrary, i.e. that M_{22} degenerate, then there exists $\alpha_0 \in R^r$, $\alpha_0 \neq 0$, such that $\alpha_0^\tau M_{22}\alpha_0 = 0$. Thus we have

$$[0 \quad \alpha_0^\tau]M[0 \quad \alpha_0^\tau]^\tau = 0 \quad (25)$$

that is,

$$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} I_n & 0 \\ K_1 & I_r \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} I_{n_E} & 0 \\ 0 & -B_2 \\ 0 & I_r \end{bmatrix}\begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} = 0.$$

It follows that

$$QG\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} = 0, \quad \alpha_0 = K_1 G\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix}. \quad (26)$$

Letting

$$\beta = G\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} \quad (27)$$

and noticing that $Q = C^\tau C$, then we obtain

$$C\beta = 0. \quad (28)$$

From (26)–(27), (18)–(20), it follows that

$$F(sE - A)\beta = F(sE - (A + BK_1))G\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} + FBK_1\beta$$

$$= \begin{bmatrix} sI_{n_E} - A_1 & 0 \\ 0 & -I_{n-n_E} \end{bmatrix}\begin{bmatrix} 0 \\ B_2\alpha_0 \end{bmatrix} + FB\alpha_0 = \begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix}. \quad (29)$$

Letting

$$\alpha = G\begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix} \quad (30)$$

and noticing (29) and (18), we have

$$(sE - A)\beta = F^{-1}\begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix}$$

$$= F^{-1}\left\{FEG + \begin{bmatrix} 0 & 0 \\ 0 & I_{n-n_E} \end{bmatrix}\right\}\begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix}$$

$$= EG\begin{bmatrix} B_1\alpha_0 \\ 0 \end{bmatrix} = E\alpha. \quad (31)$$

Combining (28) with (31) gives

$$\begin{bmatrix} C \\ sE - A \end{bmatrix}\beta = \begin{bmatrix} 0 \\ E\alpha \end{bmatrix}, \quad \text{for any } s \in C. \quad (32)$$

Since system Σ is strongly detectable by assumption, applying Lemma 3 to (32), we obtain $\beta = 0$. Therefore, $\alpha_0 = 0$ from (26)–(27). This contradicts the hypothesis. It follows that M_{22} is positive definite. Now cost (24) can be rewritten as the form

$$J(u, x(0^-)) = \int_{0^-}^{\infty} \begin{bmatrix} x_1 \\ M_{22}^{-1}M_{12}^{\tau}x_1 + v \end{bmatrix}^{\tau} \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{12}^{\tau} & 0 \\ 0 & M_{22} \end{bmatrix} \times \begin{bmatrix} x_1 \\ M_{22}^{-1}M_{12}^{\tau}x_1 + v \end{bmatrix} dt = \int_{0^-}^{\infty} (x_1^{\tau}M_{11}^{*}x_1 + w^{\tau}M_{22}w)\, dt \tag{33}$$

where

$$w(t) = M_{22}^{-1}M_{12}^{\tau}x_1(t) + v(t) \tag{34}$$

$$M_{11}^{*} = M_{11} - M_{12}M_{22}^{-1}M_{12}^{\tau} \tag{35}$$

and M_{11}^{*} is positive semi-definite as M is positive semi-definite. Since $J(u, x(0^-)) < \infty$ for any $u \in \mathcal{U}$ and M_{22} is positive definite, we see that $w(t) \in L^2$. Hence $w(t)$ is neither a Dirac-δ function nor its derivatives. Furthermore, since the trajectory $x(t)$ is a linear combination of a Dirac-δ function, its derivatives and piecewise sufficiently smooth functions (Cobb, 1981), we can see that $w(t)$ is piecewise sufficiently smooth. Paying attention to (15)–(17) and (34), we have

$$\dot{x}_1 = (A_1 - B_1M_{22}^{-1}M_{12}^{\tau})x_1 + B_1w, \quad t \geq 0 \tag{36}$$

$$0 = x_2 - B_2M_{22}^{-1}M_{12}^{\tau}x_1 + B_2w, \quad t > 0. \tag{37}$$

Hence $\bar{x}(t)$, then $x(t)$, are also piecewise sufficiently smooth.

(c) *Equivalence of the LQ problems*

Denote by $\mathcal{P}_1$ and $\mathcal{P}_2$ the LQ problems, respectively, for system (1) with cost (2) and for linear system (36) with cost

$$J(w, x_1(0)) = \int_0^{\infty} (x_1^{\tau}M_{11}^{*}x_1 + w^{\tau}M_{22}w)\, dt \tag{38}$$

where $w \in \mathcal{W}$, and $\mathcal{W}$ is the set of piecewise sufficiently smooth functions which make cost (38) finite. Consider $\mathcal{P}_2$ with the initial condition

$$x_1(0) = [I_{n_E} \quad 0]G^{-1}x(0^-). \tag{39}$$

In view of (21), (34) and

$$x_1(t) = [I_{n_E} \quad 0]G^{-1}x(t), \tag{40}$$

the solution $x(t)$ of system (1) driven by an admissible control $u \in \mathcal{U}$ with initial state $x(0^-)$ gives an admissible control $w \in \mathcal{W}$ and a solution $x_1(t)$ of system (36) driven by w with initial state (39). Combining (33) with (38) and noticing that $w(t)$ is piecewise sufficiently smooth, we conclude that

$$J(w, x_1(0)) = J(u, x(0^-)). \tag{41}$$

Hence

$$\min_{u \in \mathcal{U}} J(u, x(0^-)) \geq \min_{w \in \mathcal{W}} J(w, x_1(0)). \tag{42}$$

Inversely, if the solution $x_1(t)$ of system (36) driven by $w \in \mathcal{W}$ with initial state (39) is given, then we can obtain a control function $u(t)$, and a solution $x(t)$ of system (1) driven by this control function with initial state $x(0^-)$ by using the following relations:

$$x(t) = G\bar{x}(t) = G\begin{bmatrix} x_1(t) \\ -B_2(w(t) - M_{22}^{-1}M_{12}^{\tau}x_1(t)) \end{bmatrix}, \quad t > 0 \tag{43}$$

and

$$u(t) = w(t) - M_{22}^{-1}M_{12}^{\tau}x_1(t) + K_1x(t). \tag{44}$$

Here

$$x(0^+) = G\begin{bmatrix} x_1(0) \\ -B_2(w(0) - M_{22}^{-1}M_{12}^{\tau}x_1(0)) \end{bmatrix}. \tag{45}$$

$x(0^+)$ does not equal $x(0^-)$ in general, and

$$x(0^+) - x(0^-) = G\begin{bmatrix} 0 \\ -B_2(w(0) - M_{22}^{-1}M_{12}^{\tau}x_1(0)) - [0 \quad I_{n-n_E}]G^{-1}x(0^-) \end{bmatrix}. \tag{46}$$

A jump may occur to $x(t)$ when t moves from $t = 0^-$ to 0^+; this is because $x(0^-)$ may be inconsistent with $u(t)$. However, $x(t)$ does not have any impulsive behaviour. Hence

$$\int_{0^-}^{0^+} (x^{\tau}Qx + u^{\tau}Ru)\, dt = 0 \tag{47}$$

and

$$J(u, x(0^-)) = J(w, x_1(0)). \tag{48}$$

Equation (48) shows that $u(t)$ is an admissible control, i.e. $u(t) \in \mathcal{U}$. Thus we have

$$\min_{w \in \mathcal{W}} J(w, x_1(0)) \geq \min_{u \in \mathcal{U}} J(u, x(0^-)). \tag{49}$$

The equivalence of $\mathcal{P}_1$ and $\mathcal{P}_2$ now has been proved by (42) and (49), i.e.

$$\min_{u \in \mathcal{U}} J(u, x(0^-)) = \min_{w \in \mathcal{W}} J(w, x_1(0)) \tag{50}$$

where $x_1(0)$ is defined as in (39). Obviously, problem $\mathcal{P}_2$ is much easier to solve.

(d) *Solution of problem* $\mathcal{P}_2$

Let

$$\Sigma_2 : \begin{cases} \dot{x}_1 = (A_1 - B_1M_{22}^{-1}M_{12}^{\tau})x_1 + B_1w, \\ y = C_1x_1 \end{cases} \tag{51}$$

where C_1 is defined as in $M_{11}^{*} = C_1^{\tau}C_1$. It is clear that stabilizability of system Σ implies that of Σ_2. We show that system Σ_2 is detectable below.

Assume the contrary, i.e. that Σ_2 is not detectable, then there exist a complex s_0 with a non-negative real part, and a nonzero vector $\alpha_1 \in R^{n_E}$ such that

$$\begin{bmatrix} C_1 \\ s_0I_{n_E} - (A_1 - B_1M_{22}^{-1}M_{12}^{\tau}) \end{bmatrix}\alpha_1 = 0. \tag{52}$$

Thus

$$\begin{bmatrix} M_{11}^{*} & 0 \\ 0 & M_{22} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} = 0 \tag{53}$$

and

$$(s_0I_{n_E} - A_1)\alpha_1 = -B_1M_{22}^{-1}M_{12}^{\tau}\alpha_1. \tag{54}$$

Putting

$$\beta_1 = G\begin{bmatrix} I_{n_E} \\ B_2M_{22}^{-1}M_{12}^{\tau} \end{bmatrix}\alpha_1 \tag{55}$$

from (53), (35) and (22)–(23), we have

$$Q\beta_1 = 0 \tag{56}$$

and

$$K_1\beta_1 = M_{22}^{-1}M_{12}^{\tau}\alpha_1. \tag{57}$$

From (18)–(20), (54)–(55) and (57), we obtain

$$\begin{aligned}(s_0E - A)\beta_1 &= F^{-1}\Big\{F(s_0E - (A + BK_1))G \times \begin{bmatrix} I_{n_E} \\ B_2M_{22}^{-1}M_{12}^{\tau} \end{bmatrix}\alpha_1 + FBK_1\beta_1\Big\} \\ &= F^{-1}\Big\{\begin{bmatrix} s_0I_{n_E} - A_1 & 0 \\ 0 & -I_{n-n_E} \end{bmatrix} \times \begin{bmatrix} I_{n_E} \\ B_2M_{22}^{-1}M_{12}^{\tau} \end{bmatrix}\alpha_1 + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}K_1\beta_1\Big\} = 0. \end{aligned} \tag{58}$$

Combining (58) with (56), and noticing that $Q = C^{\tau}C$, we can see that

$$\begin{bmatrix} C \\ s_0E - A \end{bmatrix}\beta_1 = 0.$$

Since system Σ is assumed to be detectable, $\beta_1 = 0$, and then equation (55) gives $\alpha_1 = 0$. This contradicts the hypothesis. Hence system Σ_2 is detectable.

We now give the solution to the LQ problem $\mathscr{P}_2$. Since system Σ_2 is stabilizable and detectable, the LQ problem for system (36) with cost (38) has a unique solution (Kučera, 1972; Kwakernaak and Sivan, 1972):

$$w^*(t) = -M_{22}^{-1}B_1^{\tau}Px_1(t) \quad (59)$$

where P is the unique positive semi-definite matrix satisfying the Riccati equation:

$$P(A_1 - B_1M_{22}^{-1}M_{12}^{\tau}) + (A_1 - B_1M_{22}^{-1}M_{12}^{\tau})^{\tau}P - PB_1M_{22}^{-1}B_1^{\tau}P + M_{11}^* = 0. \quad (60)$$

The optimal closed loop system is asymptotically stable, and the value of the optimal cost functional is

$$J(w^*, x_1(0)) = x_1^{\tau}(0)Px_1(0). \quad (61)$$

(e) *Solution of problem* $\mathscr{P}_1$

It has been shown that $\mathscr{P}_1$ and $\mathscr{P}_2$ defined in (c) are equivalent. Therefore, from (43) and (44) the solution of the LQ problem for system (1) can be derived and written as

$$u^* = (K_1 - M_{22}^{-1}(B_1^{\tau}P + M_{12}^{\tau})[I_{n_E} \quad 0]G^{-1})x, \quad t > 0. \quad (62)$$

This is the state feedback with a constant gain. It should be mentioned that (62) holds only for $t > 0$, but not for $t \geq 0$ in general. Driven by u^*, the state $x(t)$ jumps from $x(0^-)$ to $x(0^+)$ at first, and then operates as a closed loop by control law (62). It is not hard to verify the asymptotical stability of the optimal closed loop system. The minimal cost is given as

$$J(u^*, x(0^-)) = [[I_{n_E} \quad 0]G^{-1}x(0^-)]^{\tau}P[I_{n_E} \quad 0]G^{-1}x(0^-). \quad (63)$$

Example. Given a system described by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \quad (64)$$

with initial state $x(0^-) = [1 \quad 1]^{\tau}$. The cost functional is

$$J = \int_{0^-}^{\infty} (x_1^2 + u^2)\, dt. \quad (65)$$

Obviously, system (64) satisfies all conditions required in the theorem. Following the procedure developed above, we can find the optimal control, the trajectory, and the minimal cost as follows

$$u^* = -\exp(-t); \quad x_1^* = -\exp(-t), \quad x_2^* = \exp(-t), \quad t > 0 \quad (66)$$

and

$$J_{\min} = 1. \quad (67)$$

The above representation of the optimal control can be easily written in a state feedback form, when $t > 0$. Notice that a jump occurs to $x(t)$ when t moves from $t = 0^-$ to $t = 0^+$.

3. *Conclusion*

The contribution of this paper is to establish the equivalence of the LQ problems between system (1) and a regular system subjected to the strong stabilizability and strong detectability of system (1), and then to derive the optimal control of the LQ problem for the generalized state-space system. It is clear that the result developed here can be applied to the optimal tracking problem for generalized state-space systems. Also, the authors are hopeful about the potential possibility of its application to stochastic optimal control with quadratic functional and optimal state filtering.

References

Bender, D. J. and A. J. Laub (1987a). The linear-quadratic optimal regulator for descriptor systems: discrete-time case. *Automatica,* **23,** 71–85.

Bender, D. J. and A. J. Laub (1987b). The linear quadratic optimal regulator for descriptor systems. *IEEE Trans. Aut. Control,* **AC-32,** 672–688.

Campbell, S. L. (1980). *Singular Systems of Differential Equations.* Pitman, New York.

Cobb, D. (1981). Feedback and pole placement in descriptor variable systems. *Int. J. Control,* **33,** 1135–1146.

Cobb, D. (1983a). Descriptor variable systems and optimal state regulation. *IEEE Trans. Aut. Control,* **AC-28,** 601–611.

Cobb, D. (1983b). A further interpretation of inconsistent initial conditions in descriptor variable systems. *IEEE Trans. Aut. Control,* **AC-28,** 920–922.

Cobb, D. (1984). Controllability, observability and duality in singular systems. *IEEE Trans. Aut. Control,* **AC-29,** 1076–1082.

Kučera, V. (1972). A contribution to matrix quadratic equations. *IEEE Trans. Aut. Control,* **AC-17,** 344–347.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems,* pp. 281–318. Wiley-Interscience.

Lewis, F. L. (1986). A survey of linear singular systems. *Ccts, Systems, and Signal Process,* **5,** 3–36.

Pandolfi, L. (1981). On the regulator problem for linear degenerate control systems. *J. Optimiz. Theory Applic.,* **33,** 241–254.

Rosenbrock, H. H. (1974). Structural properties of linear dynamical systems. *Int. J. Control,* **20,** 191–202.

Verghese, G. C., B. C. Levy and T. Kailath (1981). A generalized state-space for singular systems. *IEEE Trans. Aut. Control,* **AC-26,** 811–831.

Verghese, G. C., P. Van Dooren and T. Kailath (1979). Properties of the system matrix of a generalized state-space system. *Int. J. Control,* **30,** 235–243.

Yip, E. L. and R. F. Sincovec (1981). Solvability, controllability and observability of continuous descriptor systems. *IEEE Trans. Aut. Control,* **AC-26,** 702–707.