**The Optimal Regulation of Generalized State-space Systems with Quadratic Cost**

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**Abstract**—In this paper, the optimal feedback control for regulating generalized state-space systems with quadratic cost is presented by using an algebraic method, and the weighting matrix $Q$ in the cost is allowed to be positive semi-definite.

1. **Introduction**

Since Rosenbrock (1974) introduced the restricted system equivalence (RSE) for the generalized state-space system, i.e. for the linear system

$$
\dot{x} = Ax + Bu
$$

with $E$ being singular and det $(sE - A) \neq 0$, there has been a lot of research into various problems such as controllability and observability of the system (Campbell, 1980; Yip and Sincovec, 1981; Verghese et al., 1981; Cobb, 1984; also see Lewis, 1986), poles assignment and the elimination of impulsive behavior of the system by state feedback (Cobb, 1981), and optimal regulators with quadratic cost, i.e. the LQ problem (Pandolfi, 1981; Cobb, 1983; Bender and Laub, 1987a, b), and so on. However, the weighting matrices in the LQ problem investigated by Cobb (1983a) are positive definite. In this paper, the LQ problem is treated algebraically for the general case where $Q$ is allowed to be positive semi-definite, and the problem is transformed into the LQ problem for a regular state-space system by invoking strong stabilizability and strong detectability of system (1). We believe that the algebraic method adopted here is easier than the geometric one used by Cobb (1983a) for engineers to comprehend. The same problem has also been considered by Bender and Laub (1987a, b). They used Hamiltonian minimization. However, for the infinite-horizon case, stronger conditions are required there than that required in this paper.

2. **Optimal regulator with quadratic cost**

For system (1), with initial state $x(0^-)$ which can be consistent or not (Cobb, 1983b), consider the cost functional

$$
J(u,x(0^-)) = \int_0^\infty (x^TQx + u^TRu) dt
$$

where $u$ is the input, $u \in R^n$, $x$ is the state, $x \in R^n$, $Q$ and $R$ are constant matrices, $Q$ is positive semi-definite and $R$ is positive definite. Let $U$ be the set of admissible controls in which any admissible control is piecewise sufficiently smooth and makes $J(u,x(0^-))$ finite. Then the LQ problem is to find the optimal control $u^* \in U$, such that

$$
J(u^*,x(0^-)) = \min_{u \in U} J(u,x(0^-)).
$$

Definition 1. System (1) is strongly stabilizable if

$$
\text{rank } [sE - A] = n
$$

for any complex $s$ with non-negative real part and $s = \infty$.

Lemma 1. Let $T$ be a non-singular matrix such that

$$
T[sE - A] = \begin{bmatrix}
E_1 & A_2
\end{bmatrix} \begin{bmatrix}
A_1 & B_2
\end{bmatrix}
$$

where $E_1$ has full row rank; then equation (4) holds at $s = \infty$ if and only if

$$
\text{rank } \begin{bmatrix}
E_1 & 0
\end{bmatrix} = n.
$$

The proof of the lemma was given by Verghese et al. (1979). Equation (4), holding at $s = \infty$, is a necessary and sufficient condition for eliminating impulsive behaviour of system (1) (Cobb, 1981).

Lemma 2. For system (1), there exists a state feedback matrix $K \in R^{n \times n}$ such that the system

$$
\dot{x} = (A + BK)x + Bu
$$

is RSE to the system

$$
\begin{cases}
\dot{x}_1 = A_1x_1 + B_1v
\vspace{0.5cm}
0 = x_2 + B_2v
\end{cases}
$$

if and only if equation (4) holds at $s = \infty$, where $x_1 \in R^{n_1}$, $n_1 = \text{rank } E$, $x_2 \in R^{n_2 - n_1}$, $A_1 \in R^{n_1 \times n_1}$, $B_1 \in R^{n_1 \times n_2}$, $B_2 \in R^{n_2 - n_1 \times n_2}$.

By saying that system (7) is RSE to system (8), we mean that there exist two non-singular matrices $F, G \in R^{n \times n}$ such that

$$
F(sE - (A + BK)) B = \begin{bmatrix}
G & 0
\end{bmatrix} \begin{bmatrix}
0 & I_{n_2 - n_1}
\end{bmatrix} = \begin{bmatrix}
0 & -J_{n_2 - n_1}
\end{bmatrix}
$$

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Definition 2. The system
\[
\Sigma_0: \begin{cases} 
E \dot{x} = Ax + Bu \\
y = Cx 
\end{cases} \tag{9}
\]
is strongly detectable if
\[
\text{rank} \begin{bmatrix} C \\ sE - A \end{bmatrix} = n \tag{10}
\]
for any complex \( s \) with non-negative real part and \( s = \infty \). A criterion of testing that equation (10) holds at \( s = \infty \) was given by Verghese et al. (1981); see Lemma 3.

Lemma 3. Equation (10) holds at \( s = \infty \) for system \( \Sigma_0 \) if and only if there are no constant vectors \( a, b \in \mathbb{R}^n, \beta \neq 0 \), such that
\[
\begin{bmatrix} C \\ sE - A \end{bmatrix} \beta = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}
\]
for any \( s \in \mathbb{C} \).

Decompose matrix \( Q \) of cost (2) in different ways:
\[ Q = C^T C = C_1^T C_1 \]
denote
\[
\Sigma_1: \begin{cases} 
E \dot{x} = Ax + Bu \\
y = C_1 x 
\end{cases} \tag{12}
\]
\[
\Sigma_1': \begin{cases} 
E \dot{x} = AX + Bu \\
y = C_1 x 
\end{cases} \tag{13}
\]
Then we have the following lemma.

Lemma 4. \( \Sigma \) is strongly detectable if and only if \( \Sigma_1 \) is.

The lemma shows that the strong detectability for system \( \Sigma \) is independent of the decomposition of matrix \( Q \). We omit its proof and proceed directly to the LQ problem for system (1).

Theorem. If system \( \Sigma \) is strongly stabilizable and strongly detectable, then the LQ problem for system (1) has a solution, the optimal control is realized by a linear state feedback when \( t > 0 \), and the optimal closed loop system is asymptotically stable.

Proof. The basic idea of the proof is to transfer the LQ problem of the generalized state-space system into that of a regular system by means of concepts of strong stabilizibility and strong detectability.

(a) Reforming \( J(u, x(0^-)) \)

In view of the strong stabilizibility of system \( \Sigma \), from Lemma 2 we can find a matrix \( K_1 \in \mathbb{R}^{n \times n} \) such that the system
\[
E \dot{x} = (A + BK_1)x + BV \tag{14}
\]
is RSE to the system
\[
\begin{align*}
\dot{x}_1 &= A_1 x_1 + B_1 v \\
0 &= x_2 + B_2 v
\end{align*} \tag{15}
\]
i.e. there exist non-singular matrices \( F, G \in \mathbb{R}^{n \times n} \) such that
\[
x = G \ddot{x} \tag{17}
\]
\[
FEG = \begin{bmatrix} I_{n_k} & 0 \\ 0 & 0 \end{bmatrix} \tag{18}
\]
\[
F(A + BK_1)G = \begin{bmatrix} A_1 & 0 \\ I_{n-n_k} & 0 \end{bmatrix} \tag{19}
\]
\[
FB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{20}
\]
where \( n_k = \text{rank} E, \ B_1 \in \mathbb{R}^{n \times n}, B_2 \in \mathbb{R}^{(n-n_k) \times n}, A_1 \in \mathbb{R}^{n \times n}, \ i = [\begin{bmatrix} I_{n_k} \\ 0 \end{bmatrix}], \ x_1 \in \mathbb{R}^{n_k}, x_2 \in \mathbb{R}^{n-n_k}. \) Let \( x(t) \) be the solution of system (1) driven by \( u \in \mathcal{U} \) with initial state \( x(0^-) \). Set
\[
u(t) = u(t) - K_1 x(t). \tag{21}
\]
Then \( x(t) \) satisfies (14), since \( \ddot{x}(t) = G^{-1} \ddot{x}(t) \), then components \( x_1(t) \) and \( x_2(t) \) of \( \ddot{x}(t) \) satisfy (15) and (16), respectively. Rewriting \( J(u, x(0^-)) \) via (16)–(21), we have
\[
J(u, x(0^-)) = \int_{0}^{\infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \begin{bmatrix} I_{n_k} & 0 \\ 0 & -B_2 \end{bmatrix} \begin{bmatrix} G^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} L_n & K_1^T \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt. \tag{22}
\]
Representing the weighting matrix in (22) by
\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \tag{23}
\]
where \( M_{11} \in \mathbb{R}^{n \times n}, M_{12} \in \mathbb{R}^{n \times (n-n_k)}, M_{22} \in \mathbb{R}^{(n-n_k) \times (n-n_k)}, \) we have
\[
J(u, x(0^-)) = \int_{0}^{\infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt. \tag{24}
\]
(b) Showing \( M_{22} \) positive definite and rewriting cost (24).

Assume the contrary, i.e., that \( M_{22} \) degenerate, then there exists \( a_0 \in \mathbb{R}, \ a_0 \neq 0, \) such that \( a_0 M_{22} a_0 = 0 \). Thus we have
\[
[0 \ a_0] [a_0' \ a_0] = 0 \tag{25}
\]
that is,
\[
\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} K_1 \begin{bmatrix} 0 & G \\ 0 & L \end{bmatrix} \begin{bmatrix} I_{n_k} & 0 \\ 0 & -B_2 \end{bmatrix} K_1^T \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0. \tag{26}
\]
It follows that
\[
QG [B_2 a_0] = 0, \quad a_0 = K_1 G \begin{bmatrix} a_0' \\ B_2 a_0 \end{bmatrix}. \tag{27}
\]
Letting
\[
\beta = G \begin{bmatrix} 0 \\ B_2 a_0 \end{bmatrix} \tag{28}
\]
and noticing that \( Q = C^T C \), then we obtain
\[
C \beta = 0. \tag{29}
\]
From (26)–(27), (18)–(20), it follows that
\[
(sE - A) \beta = (sE - (A + BK_1)) \begin{bmatrix} B_2 a_0 \\ 0 \end{bmatrix} + FBK_1 \beta \tag{30}
\]
\[
= \begin{bmatrix} sI_{n_k} - A_1 & 0 \\ 0 & -L_{n-n_k} \end{bmatrix} \begin{bmatrix} B_1 a_0 \\ B_2 a_0 \end{bmatrix} + FBK_1 \beta = \begin{bmatrix} B_1 a_0 \\ B_2 a_0 \end{bmatrix}. \tag{31}
\]
Letting
\[
\alpha = G \begin{bmatrix} B_1 a_0 \\ 0 \end{bmatrix} \tag{32}
\]
and noticing (29) and (18), we have
\[
(sE - A) \beta = F^{-1} \begin{bmatrix} B_1 a_0 \\ 0 \end{bmatrix} \tag{33}
\]
\[
= F^{-1} \begin{bmatrix} FEG + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} \begin{bmatrix} B_1 a_0 \\ 0 \end{bmatrix} \tag{34}
\]
\[
= E \begin{bmatrix} B_1 a_0 \\ 0 \end{bmatrix} = E \alpha. \tag{35}
\]
Combining (28) with (31) gives
\[
\begin{bmatrix} C \\ sE - A \end{bmatrix} \beta = \begin{bmatrix} 0 \\ E \alpha \end{bmatrix}, \quad \text{for any } s \in \mathbb{C}. \tag{36}
\]
Since system \( \Sigma \) is strongly detectable by assumption, applying Lemma 3 to (36), we obtain \( \beta = 0 \). Therefore, \( a_0 = 0 \) from (26)–(27). This contradicts the hypothesis. It follows that \( M_{22} \) is positive definite. Now cost (24) can be rewritten as the
A jump may occur to \(x(t)\) when \(t\) moves from \(t = 0^-\) to \(0^+\); this is because \(x(0^-)\) may be inconsistent with \(u(t)\). However, \(x(t)\) does not have any impulsive behaviour. Hence

\[
\int_{0^-}^{0^+} (x^rQx + u^rRx) \, dt = 0
\]

and

\[
J(u, x(0^-)) = J(w, x_1(0)).
\]

Equation (48) shows that \(u(t)\) is an admissible control, i.e. \(u(t) \in \mathcal{U}\). Thus we have

\[
\min_{w \in \mathcal{W}} J(w, x_1(0)) = \min_{u \in \mathcal{U}} J(u, x(0^-)).
\]

The equivalence of \(\mathcal{P}_1\) and \(\mathcal{P}_2\) now has been proved by (42) and (49), i.e.

\[
\min_{w \in \mathcal{W}} J(w, x_1(0)) = \min_{u \in \mathcal{U}} J(u, x(0^-)).
\]

where \(x_1(0)\) is defined as in (39). Obviously, problem \(\mathcal{P}_2\) is much easier to solve.

(d) Solution of problem \(\mathcal{P}_2\)

Let

\[
\Sigma_2: \begin{cases}
\dot{x}_1 = (A_1 - B_1 M^{\dagger}_{12} M^{\dagger}_{12} x_1) + B_1 w, \\
y = C_1 x_1
\end{cases}
\]

where \(C_1\) is defined as in \(M^{\dagger}_{12} = C_1 C\). It is clear that stabilizability of system \(\Sigma\) implies that of \(\Sigma_2\). We show that system \(\Sigma_2\) is detectable below.

Assume the contrary, i.e. that \(\Sigma_2\) is not detectable, then there exist a complex \(\delta\) with a non-negative real part, and a nonzero vector \(\alpha_1 \in \mathbb{R}^n\) such that

\[
\left[ I_{n \times n} - (A_1 - B_1 M^{\dagger}_{12} M^{\dagger}_{12} \alpha_1) \right] \alpha_1 = 0.
\]

Thus

\[
M^{\dagger}_{12} \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \end{bmatrix} = 0
\]

and

\[
A_1 \alpha_1 = -B_1 M^{\dagger}_{12} \alpha_1.
\]

Putting

\[
\beta_1 = G \begin{bmatrix} I_{n \times n} \\ B_1 M^{\dagger}_{12} \end{bmatrix} \alpha_1
\]

from (53), (35) and (22)–(23), we have

\[
Q \beta_1 = 0
\]

and

\[
K \beta_1 = M^{\dagger}_{12} \alpha_1.
\]

From (18)–(20), (54)–(55) and (57), we obtain

\[
(sI - A - B_1) \beta_1 = F^{-1} \left[ F(x_k; E - (A + BK)) G \right]
\]

and

\[
\beta_1 = \begin{bmatrix} I_{n \times n} \\ B_1 M^{\dagger}_{12} \end{bmatrix} \alpha_1
\]

Combining (58) with (56), and noticing that \(Q = C^TC\), we can see that

\[
\begin{bmatrix} C \\ sI - A \end{bmatrix} \beta_1 = 0.
\]

Since system \(\Sigma\) is assumed to be detectable, \(\beta_1 = 0\), and then equation (55) gives \(\alpha_1 = 0\). This contradicts the hypothesis. Hence system \(\Sigma_2\) is detectable.
We now give the solution to the LQ problem $\mathcal{P}_1$. Since system $\mathcal{S}_2$ is stabilizable and detectable, the LQ problem for system (36) with cost (38) has a unique solution (Kwakernaak and Sivan, 1972):

$$w^*(t) = -M_{22}^{-1} B_1^TPx(t)$$ (59)

where $P$ is the unique positive semi-definite matrix satisfying the Riccati equation:

$$P(A_1 - B_1M_{22}^{-1}M_{12}) + (A_1 - B_1M_{22}^{-1}M_{12})^TP - PB_1M_{22}^{-1}B_1^TP + M_{11} = 0.$$ (60)

The optimal closed loop system is asymptotically stable, and the value of the optimal cost functional is

$$J(w^*, x_1(0)) = x_1^T(0)Px_1(0).$$ (61)

(c) Solution of problem $\mathcal{P}_2$

It has been shown that $\mathcal{P}_1$ and $\mathcal{P}_2$ defined in (c) are equivalent. Therefore, from (43) and (44) the solution of the LQ problem for system (1) can be derived and written as

$$u^* = (K_1 - M_{22}^{-1}(B_1^TP + M_{12})[I_{n_2} 0]G^{-1})x, \quad t > 0.$$ (62)

This is the state feedback with a constant gain. It should be mentioned that (62) holds only for $t > 0$, but not for $t = 0$ in general. Driven by $u^*$, the state $x(t)$ jumps from $x(0^-)$ to $x(0^+)$ at first, and then operates as a closed loop by control law (62). It is not hard to verify the asymptotical stability of the optimal closed loop system. The minimal cost is given as

$$J(u^*, x(0^-)) = ([I_{n_2} 0]G^{-1}x(0^-))^TP[I_{n_2} 0]G^{-1}x(0^-).$$ (63)

Example. Given a system described by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x}_1 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{x}_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$ (64)

with initial state $x(0^-) = [1 \ 1]^T$. The cost functional is

$$J = \int_{0^-}^{t} (x_1^2 + u^2) \, dt.$$ (65)

Obviously, system (64) satisfies all conditions required in the theorem. Following the procedure developed above, we can find the optimal control, the trajectory, and the minimal cost as follows

$$u^* = -\exp(-t); \quad x_1^* = -\exp(-t),$$

$$x_2^* = \exp(-t), \quad t > 0$$ (66)

and

$$J_{\text{min}} = 1.$$ (67)

The above representation of the optimal control can be easily written in a state feedback form, when $t > 0$. Notice that a jump occurs to $x(t)$ when $t$ moves from $t = 0^-$ to $t = 0^+$.  

3. Conclusion

The contribution of this paper is to establish the equivalence of the LQ problems between system (1) and a regular system subjected to the strong stabilizability and strong detectability of system (1), and then to derive the optimal control of the LQ problem for the generalized state-space system. It is clear that the result developed here can be applied to the optimal tracking problem for generalized state-space systems. Also, the authors are hopeful about the potential possibility of its application to stochastic optimal control with quadratic functional and optimal state filtering.

References


